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# On Skew Group Rings (Skew Polynomial Rings, Group Rings and Related Topics)

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# On skew group rings

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Let  $R$  be a ring with  $1$ , and  $G$  a group.  $U(R)$  denotes the group of units of  $R$ . Given maps  $\alpha: G \rightarrow \text{Aut}(R)$  and  $\gamma: G \times G \rightarrow U(R)$  such that

$$(i) \quad \gamma(g, h)\gamma(gh, i) = \gamma(h, i)^{\alpha(g)^{-1}} \gamma(g, hi)$$

and

$$(ii) \quad \gamma(g, h)r^{\alpha(gh)^{-1}} = r^{\alpha(g)^{-1}}\alpha(g)^{-1} \gamma(g, h)$$

for all  $g, h, i \in G, r \in R$ , we define the crossed product  $R * G$  to be a free  $R$ -module with basis  $\{\bar{g} \mid g \in G\}$ . The multiplication is given by the rule

$$(r_g \bar{g})(r_h \bar{h}) = r_g r_h^{\alpha(g)^{-1}} \gamma(g, h) \overline{gh}.$$

This makes  $R * G$  an associative ring with unit element

$\gamma(1, 1)^{-1} \bar{1}$ . The map  $r \rightarrow r\gamma(1, 1)^{-1} \bar{1}$  is a ring monomorphism of

$R$  into  $R * G$ . We therefore consider  $R$  as a subring of  $R * G$ .

If  $\gamma(g, h) = 1$  for all  $g, h \in G$ ,  $R * G$  called a skew group ring, and if  $\alpha(g) = 1$  for all  $g \in G$ ,  $R * G$  is called a twisted group ring.

Let  $S$  be any ring. Let  $R * G$  be a crossed product with  $G$  a finite group, and  $V, V'$   $(S, R * G)$ -modules. For  $g \in G$

and  $k \in \text{Hom}_{(S, R)}(V, V')$ , we define  $k^g(v) = k(v\bar{g}^{-1})\bar{g}$  for all

$v \in V$ . One may check that  $k \rightarrow k^g$  defines a group action of

$G$  on  $\text{Hom}_{(S, R)}(V, V')$ . It is clear that the fixed submodule

is  $\text{Hom}_{(S, R * G)}(V, V')$ . Therefore  $t_G(k) = \sum_{g \in G} k^g$  is an  $(S, R * G)$

omomorphism for every  $k \in \text{Hom}_{(S,R)}(V, V')$ . If there exists an  $h \in \text{End}_{(S,R)}(V')$  such that  $t_G(h) = 1_{V'}$ , then  $\hat{k} = t_G(hk)$  is an  $(S, R^*G)$ -homomorphism and  $\hat{k} = k$  on every  $R^*G$ -submodule of  $V$ .  $C(R)$  denotes the center of  $R$ . If there exists an element  $c \in C(R)$  such that  $t_G(c) = \sum_{g \in G} c^{\alpha(g)} = 1$ , then  $t_G(T_c) = 1$ , where  $T_c \in \text{End}_{(S,R)}(V')$  denotes right multiplication by  $c$ . If  $V'$  is  $|G|$ -torsion free and  $V \cdot |G|$ , then we can define an element  $h \in \text{End}_{(S,R)}(V')$  by  $h(v) = |G|^{-1}v$  for all  $v \in V'$ . Clearly,  $t_G(h) = 1$ .

Now, the proof of the following is easy.

Proposition 1. Let  $W \subset V$  be  $(S, R^*G)$ -modules. Suppose there exists an element  $c \in C(R)$  such that  $t_G(c) = 1$ .

If  $W \not\subseteq_S V_R$ , then  $W \not\subseteq_S V_{R^*G}$ .

We note that if the order of  $G$  is invertible in  $R$ , then  $|G|^{-1} \in C(R)$  and  $t_G(|G|^{-1}) = 1$ .

A ring  $R$  is said to be fully right idempotent if every right ideal of  $R$  is idempotent. For example, von Neumann regular rings, right V-rings, and ring which are direct sum of simple rings, are fully right idempotent.

Corollary 1. Let  $R^*G$  be a crossed product with  $G$  a finite group. Suppose there exists an element  $c \in C(R)$  such that  $t_G(c) = 1$ .

(1) If  $R$  is fully right idempotent, then so is  $R^*G$ .

- (2) If  $R$  is regular, then so is  $R^*G$ .
- (3) If  $R$  is a right V-ring, then so is  $R^*G$ .
- (4) If  $R$  is a direct sum of simple rings, then so is  $R^*G$ .

Proof. We prove only (3). If  $K$  is a maximal right ideal of  $R^*G$ , then there exists a maximal submodule  $M$  of  $R^*G_R$  which contains  $K$ . It is easy to see that  $\bigcap_{g \in G} Mg = K$ . Therefore  $P = R^*G/K$  is a direct sum of simple right  $R$ -modules, and hence  $P$  is an injective  $R$ -module. Let  $E$  be an injective hull of  $P_{R^*G}$ . Since  $P \triangleleft E_R$ ,  $P = E$  by Proposition 1.

Let  $G \subset \text{Aut}(R)$  be a finite group, and  $R^*G$  a skew group ring.  $R$  can be viewed as a right  $R^*G$ -module by defining  $r \cdot \sum x_g g = \sum (rx_g) g$ ; for  $x_g$  and  $r$  in  $R$ . If we set  $f = \sum_{g \in G} g$ , then  $R \simeq fR^*G$  as right  $R^*G$ -modules. The fixed subring is denoted by  $R^G$ ;  $R^G = \{r \in R \mid r^g = r \text{ for all } g \in G\}$ . For a module  $V$  over a ring  $S$ , let  $L(V_S)$  denote the lattice of  $S$ -submodules of  $V$ .

Lemma 1. Let  $G \subset \text{Aut}(R)$  be finite. Suppose there is an element  $c \in R$  such that  $t_G(c) = 1$ . Then the following are equivalent:

- 1)  $R_{RfR}$  is s-unital; that is,  $r \in r \cdot RfR$  for all  $r \in R$ .
- 2)  $L(R^G_{(R^*G)}) \rightarrow L(R_{R^*G}); I \rightarrow IR$ , is a lattice isomorphism.

Proof. Since  $t_G(r \cdot R^*G)R = r \cdot RfR$  for every  $r \in R$ , the assertion is clear.

Corollary 2. Let  $R$  be a fully right idempotent ring, and  $G \subset \text{Aut}(R)$  finite. Suppose there is an element  $c \in C(R)$  such that  $t_G(c) = 1$ . Then the lattice of right ideals of  $R^G$  is isomorphic to the lattice of  $G$ -invariant right ideals of  $R$ .

Proof. We set  $Q = R * G$ . By the part (1) of Corollary 1,  $Q$  is fully right idempotent. Let  $r$  be an element of  $R$ . Then,  $\text{fr} \in (\text{fr} Q)^2 \subset \text{fr}(RfR)$  and so  $r \in r \cdot (RfR)$ .

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